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A mixing length model for turbulent boundary layers over rough surfaces

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Abstract—A methodology to analyze turbulent boundary layers over rough surfaces is developed. A new formulation of the mixing length model, expressed in the velocity variable, is instrumental in the modeling effort, and the surface roughness effect is embedded in an amplification factor as a multiplier to the mixing length. It ranges from unity for a smooth surface and varies upward for rough surfaces, depending on the type and magnitude of the roughness. The new method is demonstrated in the case of flow over a flat plate with two surface roughnesses. With a length Reynolds number of 10 million, the local skin friction is increased by as much as 60% when the ratio of roughness height to length is 1 : 10 000.

1. INTRODUCTION

THE INFLUENCE of surface roughness on fluid dynamics and heat transfer is an accepted phenomenon, as witnessed by the continuing stream of investigations since the early work by Nikuradse [1], on the role of surface roughness in turbulent boundary layers. Much has been accomplished, resulting in more test data, analyses and correlations—theory, basic mechanism and physical components. While the state of understanding continues to improve, the current analytical procedures are in some aspects still fragmented and in want. With the advent of computational fluid dynamics, there is a need for a unified methodology to account for the effects of surface roughness. The need is especially marked in aerospace applications where high speed can greatly magnify even a moderate surface roughness such as those encountered in re-entry cooling technology and on gas turbine blades.

Motivated by the last-mentioned application, this work was undertaken as a step toward a more consistent approach by providing a model of analyzing the turbulent boundary layers on rough surfaces. The model is based on Prandtl's mixing length, modified so as to admit a roughness-dependent amplification factor, resulting in an analytical structure different from that of the Prandtl–van Driest family. It is this amplification factor in which trends of the experimental data are reflected in a plausible way. The model formulated is shown to satisfactorily mimic the experimental features in the velocity distributions in the rough-wall boundary layers and in the pipe flow data of Nikuradse [1] and Moody [2]. So far the model is restricted to the fluid dynamic aspects, its extension to the thermal behavior is under way and will be reported separately.

2. THE MIXING LENGTH AND SURFACE ROUGHNESS

2.1. *A brief re-visit with Prandtl's analysis*

Nikuradse's rough-pipe data and those of others have established that in the wall-law region, an experimentally determined velocity distribution when non-dimensionalized by the plus-coordinates, u^+ and y^+, shows a definite, parallel downward shift‡ from the wall-law for a smooth surface. The slope is unchanged. Thus, between the two boundary layers—rough and smooth walls—there is a similarity and a difference. For clarity of discussion, a few essentials of the early developments are re-cited. We begin with the empirical wall-law for a smooth surface

$$u^+ = \frac{1}{K}\log_e y^+ + C = A_1 \log_{10} y^+ + C \qquad (1)$$

in which the constants K and C are approximately 0.4 and 5.5, respectively. This empirical law led Prandtl to his hypothesis of a mixing length l, which, in turn, gives rise to a turbulent viscosity and the turbulent shear stress. These developments are summarized by the following equations:

$$\tau = \mu_t(\partial u/\partial y), \quad \tau = \rho l^2(\partial u/\partial y), \quad l = Ky. \qquad (2a\text{–}c)$$

The ability of Prandtl's mixing length model in recovering the wall-law is an essential attraction for its wide acceptance in the fluids engineering community. Its adoption became more widespread when van Driest [3] postulated a damping factor

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‡ The term, velocity shift, or Δu^+, shall mean the downward shift of u^+ for a rough surface from that for a smooth surface in the region where the wall-law is valid.

NOMENCLATURE

A_1	wall-law constant, equation (1)
A_v	van Driest's damping constant (= 26), equation (3)
B	Nikuradse intercept function, equation (18)
C	wall-law intercept for smooth walls, equation (1)
C_f	local surface friction coefficient, $\tau/\frac{1}{2}\rho U^2$
D	van Driest damping factor, equation (3)
E	wall-law constant, exponential form, equation (5b)
f	Darcy's pipe friction factor, $8(u_t/u_a)^2$
k	wall roughness height
k^+	roughness Reynolds number, ku_t/v
K	wall-law (von Karman) constant (0.40, 0.411)
l	mixing length
l^+	dimensionless mixing length, lu_t/v
r_0	pipe radius
r_0^+	non-dimensional pipe radius, r_0u_t/v
R	amplification factor
Re	pipe flow Reynolds number, $2u_ar_0/v$
Re_x	distance Reynolds number, xU/v
u	time-averaged turbulent velocity parallel to a wall
u_a	cross-section averaged velocity of u
u_t	friction velocity, $\sqrt{(\tau/\rho)}$
u^+	dimensionless velocity, u/u_t
Δu^+	magnitude of asymptotic downshift of u^+
U	freestream velocity
x	distance along a flat plate
y	distance from wall
y^+	dimensionless normal distance, yu_t/v.

Greek symbols

ε^+	turbulent viscosity/molecular viscosity, μ_t/μ
μ	molecular viscosity
μ_t	turbulent viscosity
v	kinematic viscosity of fluid
ρ	fluid density
τ	wall shear stress.

Subscripts

a	average
t	frictional; turbulent
w	wall.

which modifies Prandtl's mixing length, equation (2c), by his wall correction formula

$$D = 1-\exp(-y^+/A_v). \qquad (3)$$

For the damping constant A_v, he recommended a value of 26. As is well known, the model consisting of equations (2c) and (3) has led to a good accounting of the entire profile, unifying the sublayer, the buffer zone, and the fully turbulent region. Such a procedure is now well established for flows over smooth surfaces. For rough surfaces, however, the situation is not as clear.

2.2. *Recent modifications for rough surfaces*

Since surface roughness promotes mixing in the flow, a simple and, perhaps, elementary modification is to postulate that the mixing length be increased by a factor R. Trying a new formulation

$$l = RKy \qquad (4)$$

would satisfy the conceptual requirement as noted. However, its use results in a wall-law with a slope of $1/(RK)$, not $1/K$ as confirmed by the overwhelming experimental evidence, thus leading to a confict. Apparently aware of this, van Driest [3] sought to adjust his constant A_v so that it decreases as the surface roughness increases in size. By matching with the velocity shifts measured on surfaces with known roughnesses, he established a relation of his damping constant with k^+ in which it successively decreases from 26 for $k^+ = 0$ to a value of zero for $k^+ = 55$. Beyond this, van Driest's modification is not, as he stated, capable of describing the rough-wall velocity distributions.

Along the same direction, McDonald and Fish [4] let van Driest's damping factor D exceed the value of 1 when $k^+ > 55$. They did this by an additive term to the right-hand side of equation (2c) and showed that such a modification is only usable up to $k^+ = 10\,000$. Similarly, Heazler *et al.* [5] allowed the damping constant A_v to remain at zero for $k^+ > 55$, but increased the mixing length by an amount Δl^+ which depends on the overage (k^+-55).

Differing from the preceding steps, Rotta [6] proposed a coordinate shift. His procedure is shown by Cebeci and Smith [7] as identical to replacing y^+ by $(y^++\Delta y^+)$ in equation (2c), where Δy^+ is a pre-fixed coordinate shift. Conceptually speaking, Rotta's idea recognized that at the tips of the surface protrusions, the plus-velocity represents a cumulative effect emanating from the recesses of the roughness elements. More recently, Granville [8] has combined the approaches of van Driest and of Rotta, resulting in a formulation that can be used for arbitrary values of k^+—from the smooth through the transitionally rough and to the fully rough regime.

In the modifications enumerated so far, there is however a common inadequacy: that the required condition of $du^+/dy^+ = 1$ at the wall $y^+ = 0$ is not always satisfied, for the mixing length there is no longer zero but finite. In finite-difference calculations, this inconsistency may prove crucial.

2.3. *A new formulation for smooth surfaces*

Since equation (1) for the wall-law can be written as

$$y^+ = \exp(Ku^+)/E, \quad E = \exp(KC) \quad (5a, b)$$

Prandtl's non-dimensional mixing length $l^+ = (lu_\tau/v)$ can be cast, by virtue of equation (5a), as

$$l^+ = (K/E)\exp(Ku^+). \quad (6)$$

Its use in lieu of Ky^+ would lead to the law of the wall, but with an undefined integration constant which can only be fixed by integrating from the wall position, $y^+ = 0$, where it is required that l vanish. This is done by modifying equation (6) to one of the following forms, while preserving the asymptotic behavior at large y^+ or u^+:

$$l^+ = (K/E)[\exp(Ku^+) - \exp(-Ku^+)] \quad (7a)$$

$$l^+ = (K/E)[\exp(Ku^+) - [1 + (Ku^+) + (Ku^+)^2/2]] \quad (7b)$$

It is noted that at this stage of the development, the left-hand sides of equations (7a) and (7b) can just as well be replaced by ε^+, the turbulent viscosity ratio. (In fact, Spalding [9] had earlier shown a single velocity expression from which he deduced a turbulent viscosity of the type of equation (7b).)

To summarize, there are four proposed models, all quite similar in their general outline and all capable of recovering the wall-law

$$\varepsilon^+ = (\mu_t/\mu) = (K/E)[\exp(Ku^+) - [1 + (Ku^+) + (Ku^+)^2/2]] \quad (8a)$$

$$l^+ = (lu_\tau/v) = (K/E)[\exp(Ku^+) - [1 + (Ku^+) + (Ku^+)^2/2]] \quad (8b)$$

$$\varepsilon^+ = (K/E)[\exp(Ku^+) - \exp(-Ku^+)] \quad (8c)$$

$$l^+ = (K/E)[\exp(Ku^+) - \exp(-Ku^+)]. \quad (8d)$$

It is from these formulations that a further delineation shall be made as to which is to be recommended: using 0.4 for K and 8.134 for E (these values yield $C = 5.24$, van Driest's intercept), four curves of u^+ vs y^+, in addition to van Driest's curve are obtained and shown in Fig. 1. Among the results from the various models, noticeable differences occur only in the buffer region between those of models represented by equations (8b) and (8c). It may be noted in passing that the models of equations (8a) and (8c) give the following closed forms:

$$y^+ = u^+ + [\exp(Ku^+) - [1 + (Ku^+)^2/2 + (Ku^+)^3/6]]/E \quad (9)$$

$$y^+ = u^+ + (2/E)[\cos h(Ku^+) - 1]. \quad (10)$$

It appears that for smooth walls, formulations by equations (8a) and (8d) are preferable to the other two, though all four models reproduce the wall-law at large values of u^+.

2.4. *Extension to rough surface*

All of the four forms of equations (8) have one feature in common: at large values of u^+, their right-hand sides become

$$\varepsilon^+ \quad \text{or} \quad l^+ = (K/E)\exp(Ku^+) \quad (11)$$

from which the wall-law is recovered. It is significant to note that the wall-law slope is dictated by K inside the exponential term of equations (8) and that it is unaffected by the proportionate constant (K/E). It is the latter that provides an extra degree of freedom that can be used to account for the roughness influence in raising the mixing length or turbulent viscosity.

Accordingly, the right-hand sides of equations (8) are multiplied by an amplification factor R the magnitude of which depends on the roughness Reynolds number, k^+. To illustrate this method of analysis, equation (8c) is selected for the process: first, the rough-surface turbulent viscosity is, from equation (8c), represented by

$$\varepsilon^+ = R(K/E)[\exp(Ku^+) - \exp(-Ku^+)] \quad (12)$$

and the resulting velocity profile is a simple, closed expression

$$y^+ = u^+ + (2R/E)[\cos h(Ku^+) - 1]. \quad (13)$$

At large values of u^+, equation (13) is reduced to

$$u^+ = [\log_e y^+ + \log_e E - \log_e R]/K \quad (14)$$

which has a slope of $1/K$, the same as for smooth surfaces, and a last term denoting the downward shift of u^+ in the wall region. The shift and the amplification factor R are therefore related by

$$\Delta u^+ = (1/K)\log_e R. \quad (15)$$

The preceding result, though derived from using the model of equation (8c) is common to the other models as well. It is this inherent feature in all four models of equations (8) that distinguishes the present analysis from the other models discussed previously.

2.5. *Preferred model formulations*

The next step is to establish from the four models of equations (8) a clearer choice by examining how well the numerical results based on these models compared with the available experimental data over a rough surface. For this purpose, however, the only data were found in Robertson *et al.*'s work [10]. They measured the velocity downshift Δu^+ from $y^+ = 10$ to the fully developed region for different surface roughnesses. The measured values of Δu^+ were in reference to the smooth surface wall-law of equation (1) with $A_1 = C = 5.6$. Using the downshift measured

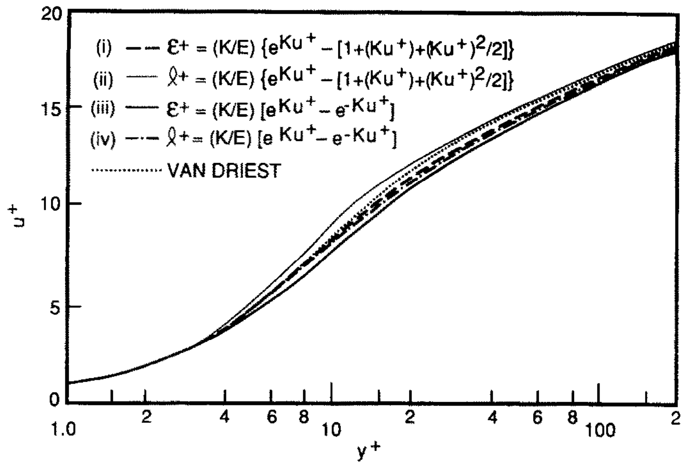

FIG. 1. Comparative velocity profiles from models ($K = 0.4$, $E = 8.134$, $C = 5.24$).

in the wall-law region to calculate an amplification factor R from equation (15), analytical velocity distributions were obtained by direct numerical integration, starting from the wall position, with the turbulent viscosity or mixing length models of equation (8). Therefore, for each model, a set of calculated velocity curves could be obtained to compare with Robertson *et al.*'s data in the range of y^+ from 10 to 200. Of the four sets, those using equations (8c) and (8d) are indistinguishable from each other and are in far better agreement with the data than the other two models. This favorable correlation is testified by Fig. 2 and led to the adoption of equation (8d) for an exponential mixing length as the candidate model of this analysis. There is no particular reason for favoring equation (8d), except that mixing length is more in use than turbulent viscosity as the modeled parameter.

3. THE ROUGHNESS AMPLIFICATION FACTOR

What is required next is a quantitative relation connecting the amplification factor R and the roughness number k^+. For a particular roughness pattern, it can only come from empirical data. An extensive survey of the literature indicated that there are only a few types for which the measured data can be used to extract the amplification factors. There are four data sources: Nikuradse's sandgrain, Moody's random roughnesses [2], Colebrook–White's mixed-sand surface [11], and Hama's wirescreen data [12]. These four sources cover two distinct classes: (i) uniform roughnesses—Nikuradse and Hama, (ii) random roughnesses—Colebrook–White and Moody. Of the flour, Nikuradse's data are more comprehensive than the others.

3.1. *Nikuradse's treatment of his roughness data*

3.1.1. *The B-function.* Starting with a smooth pipe, he first integrated equation (1) to obtain a relation between the average and the friction velocities, and, with minor adjustments, arrived at a pipe friction formula which agreed well with the experimental data. For flows in rough pipes, Nikuradse followed a similar procedure and obtained a pipe friction correlation but with an undetermined intercept constant C. Its value was fixed by comparing with his measured friction coefficient for a known surface roughness. This procedure led to a relationship between the velocity downshift Δu^+—through the intercept constant C—and the surface roughness. Reynolds number—through the pipe roughness and friction. Nikuradse chose to organize his data by means of his B-function which is defined through the expression

$$u^+ = (\log_e y^+ - \log_e k^+)/K + B. \qquad (16)$$

An explicit connection between the B-function and the intercept constant C in equation (1) can be readily obtained. Thus if C is set to 5.5 for a smooth surface, then B must assume the following limiting form as k^+ approaches zero:

$$B = 5.5 + 5.75 \log_{10} k^+. \qquad (17)$$

At the extreme of large values of k^+, experimental data established that B approaches a constant of 8.48. Hence Nikuradse's definition led to the interpretation that the downward velocity shift in the wall-law region is the B-function value of equation (17) for a smooth surface minus the B-value in equation (16) that describes the wall-law over a rough surface. Hence, for large k^+

$$u^+ = (5.5 + 5.75 \log_{10} k^+) - 8.48$$
$$= 2.5 \log_e k^+ - 2.98. \qquad (18)$$

When the roughness number is large, say $k^+ > 100$, for sandgrain roughness, B can be safely put to 8.48, then equation (18) coupled with equation (15) yields an asymptotic, linear relation between the mixing length amplification R and k^+.

3.1.2. *Nikuradse's B-function re-analyzed.* While his

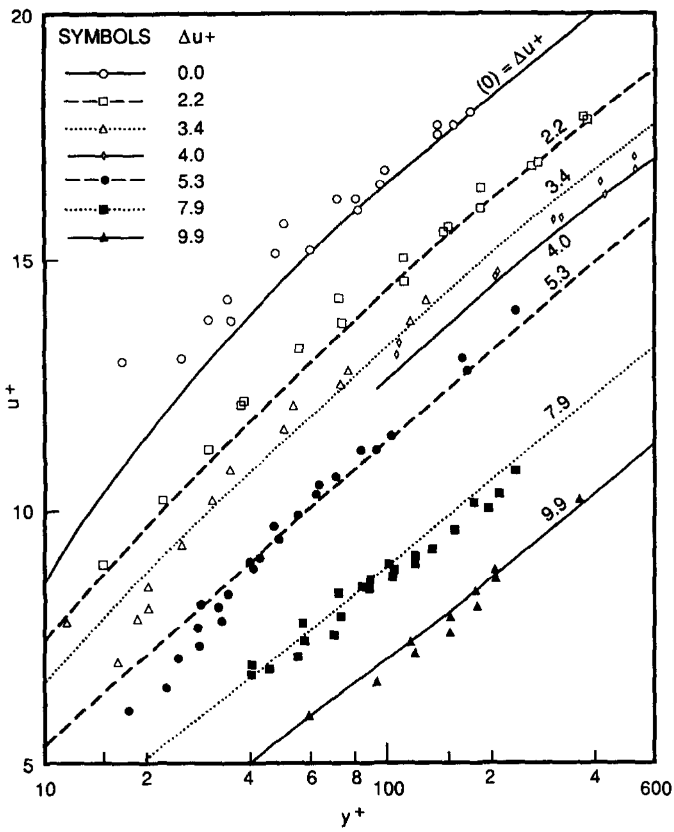

FIG. 2. Comparison of Robertson *et al.*'s data with calculated profiles.

experimental pipe friction data and his method of presentation remain undisputed, it is generally believed that there is room for improvement in his assumption of the wall-law for the pipe cross-section. Hence, starting from a more logical step we use the following equation to generate the established velocity profile:

$$[1+(l^+)^2(\mathrm{d}u^+/\mathrm{d}y^+)](\mathrm{d}u^+/\mathrm{d}y^+) = [1-(y^+/r_0^+)] \quad (19)$$

where r_0^+ is the dimensionless pipe radius in the plus coordinates, and y^+ is the distance from the pipe wall. In the equation, the mixing length is obtained from equation (8d).

The first step was to verify that the use of these two equations, equations (8d) and (19), would lead to the smooth pipe friction correlation. This was done by taking 0.4 for K, 9.025 for E, and by setting R to 1. The agreement was rather poor. Even with the widely accepted Prandtl–van Driest formula for the mixing length, the calculated friction factor vs Reynolds number relation shows an appreciable deviation from the established correlation. However, by introducing a van Driest damping factor of equation (3) into equation (8d), the result was a remarkable improvement. With this modification, rough pipe flows were analyzed essentially by cut and trial by varying R until the calculated friction values were in acceptable agreement with the data over a reasonable Reynolds number range. In fact, the process of determining the mixing length amplification R was also carried out for Moody's random roughnesses, characteristic of commercial pipes. With the amplification R thus determined, an asymptotic velocity downshift Δu^+ can be calculated from equation (15). And from Δu^+, a corresponding B-function was obtained. Results of the preceding steps are shown in the figures as follows: in Figs. 3(a) and (b) are the familiar pipe friction diagrams for Nikuradse's sandgrain roughnesses and for Moody's commercial pipes with random roughnesses. In both figures, the calculated values in the present work are the symbols. For each roughness-to-radius ratio, good agreement with the established correlations (curves) can be seen. The mixing length amplification factors for these two types of roughnesses along with those for Colebrook–White's mixed sand and Hama's wirescreens are given in Fig. 4. Of interest to observe is that the curves of R vs k^+ for Nikuradse's sandgrain and Hama's wirescreens show a low plateau value at $k^+ < 4$ and that they both have a transition region between 4 and 40 and become linear in k^+ afterwards, as discussed before. In contrast to these two patterns of uniform roughnesses, the curves for random roughnesses exhibit a continuous

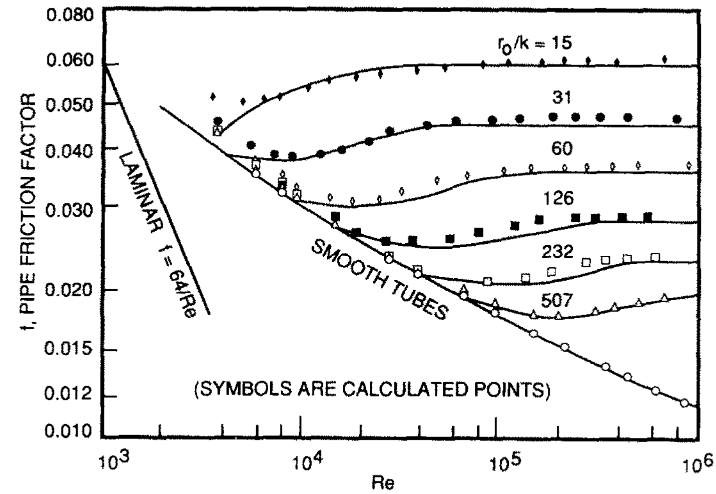

FIG. 3(a). Calculated pipe friction factors for Nikuradse's sandgrain roughnesses.

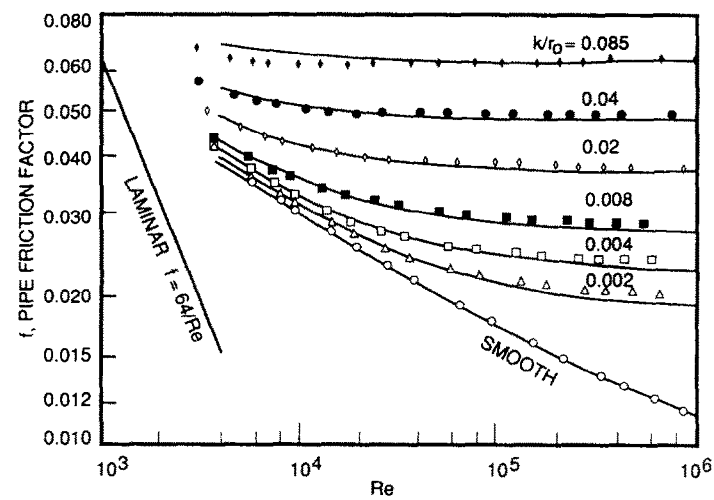

FIG. 3(b). Calculated pipe friction factor for Moody's commercial random roughnesses.

variation without the zonal features for the other type. Finally, the re-evaluated B-function and Nikuradse's values are compared in Fig. 5. It is worth noting that although the difference between them is not large, the calculated friction results from the two sources turned out to be quite significant. Corresponding to Moody's random roughness friction data, the B-function is also presented in Fig. 5.

The computed amplification factors for various values of k^+ are tabulated in Table 1 for Nikuradse's sandgrain and Moody's random roughnesses. At large values of k^+, linear relations of the following types can be used with confidence:

Nikuradse: $R = 0.3036k^+$, $k^+ > 100$

Hama: $R = 0.586k^+$, $k^+ > 30$

Moody: $R = 0.2950k^+$, $k^+ > 600$

Colebrook–White: $R = 0.327k^+$, $k^+ > 40$.

4. THE VELOCITY FROM THE WALL TO THE LAW OF THE WALL

With the R–k^+ relations established at least for the commonly encountered roughnesses, it becomes of interest to show that the present methodology can indeed produce the velocity profiles from the wall to where the wall-law is valid. (It should be noted that the question of where y^+ is referenced is not taken up in the present work. A commonly accepted reference point is about 70% of the height below the protrusion tips.) Values for y^+ were chosen between 0 and 400, and the two turbulence models given by equations (8c) and (8d) were used. As an illustration, Colebrook–White's data were used in the R–k^+ relation. The numerical constants K and E were set at 0.4 and 9.025, with the latter corresponding to an intercept value of 5.5, first used by Nikuradse. Computed curves in Fig. 6 show features consistent with experimental data: at

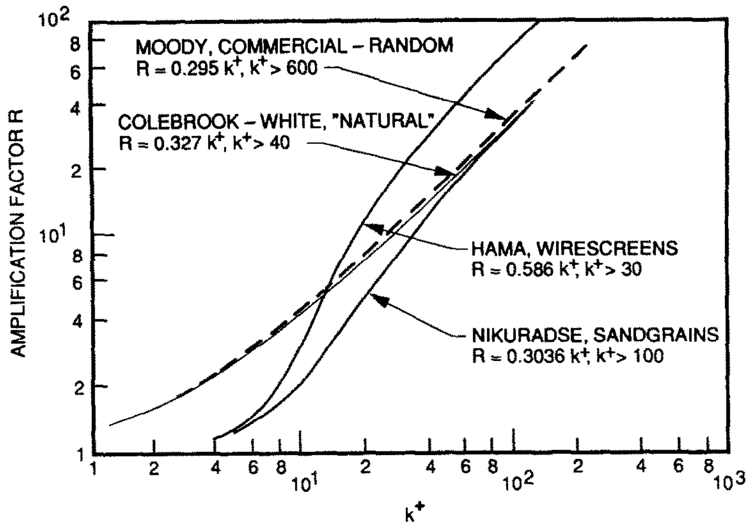

FIG. 4. Mixing length amplification vs surface roughness.

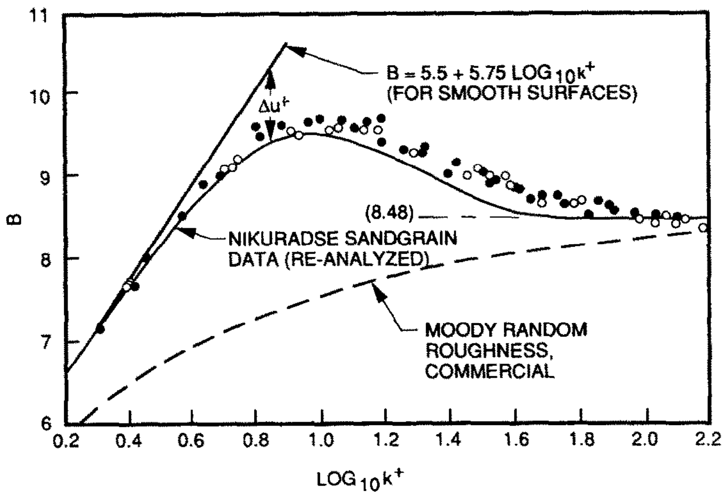

FIG. 5. Distributions of velocity-shift function B vs roughness Reynolds number.

$y^+ > 100$, all curves for different roughnesses have the same slope of $1/K$; and at $y^+ = 0$, $du^+/dy^+ = 1$ is satisfied by all curves. In addition, the laminar sublayer thickness is seen to decrease as k^+ increases, i.e. the surface becomes rougher.

5. FLOW OVER A ROUGH FLAT PLATE

It is well known that the basic Prandtl–van Driest model for the mixing length works well for flows with mild pressure gradients. Since its emergence, there have been quite a number of modifications to extend its applications to include compressibility, surface transpiration, adverse pressure gradient and other factors. With the exponential model developed in this analysis for rough surfaces, the first expectation is of course that it should work well for simple flows with surface roughness. Hence as a first demonstration, calculations were carried out for a flat plate with Nikuradse's sandgrain roughness, for which the amplification factors are taken from Table 1. Starting with turbulent flow at the leading edge and with the roughness defined by (kU/ν) of 0, 500, and 1000, boundary layer developments were tracked until the local Reynolds numbers were well into the 'established' range. The numerical constants were $K = 0.41$ and $E = 7.768$ $(C = 5)$. The eddy viscosity from equation (8d) was not to exceed Clauser's limit of 0.0168 times the displacement Reynolds number. Using a simple marching code, results of the local friction are shown in Fig. 7. As a first check, the computed friction was found to correlate well with the accepted equation for flows over a smooth surface. For the other two surface conditions, it should be observed that the increase in friction in the leading edge region with low local Reynolds numbers was

Table 1. Mixing length amplification for sandgrain (Nikuradse) and commercial random (Moody) roughnesses

k^+	Nikuradse's sandgrain	Moody's commercial random
0.0	1	1
2.0	1.0288	1.5173
4.0	1.1278	2.2534
8.0	1.6059	3.6637
10.0	2.0003	4.3767
12.0	2.4778	5.0748
16.0	3.5597	6.4233
20.0	4.7771	7.7445
25.0	6.4263	9.3921
30.0	8.1745	11.0470
40.0	11.7392	14.3558
50.0	15.0897	17.6186
60.0	18.2128	20.8224
80.0	24.2704	27.1310
100.0	30.3600†	33.3253
120.0	—	39.4392
160.0	—	51.5278
200.0	—	63.4309
400.0	—	121.4452
600.0	—	177.0†

† R becomes linear afterwards: $R = 0.3036k^+$ (Nikuradse); $R = 0.295k^+$ (Moody).

much more pronounced than that in the downstream region with high Reynolds numbers. This is due, of course, to the fact that the roughness effect exerts itself through a roughness Reynolds number, k^+, in which the local friction velocity plays an equally influential role, as does the physical size of a protrusion. Thus, in the downstream region, roughness influence becomes diluted as surface friction is reduced.

To show some finer details, calculated velocity profiles at a streamwise location of $Re_x = 10^7$ are given in Fig. 8 for the three surfaces. Overall, the velocities in the rough-wall boundary layers are, as expected, lower than in a smooth-surface layer. In the immediate vicinity of a surface, however, the reverse happens, as it should. These relative velocity distributions are clearly discernible in Fig. 8.

6. CONCLUDING REMARKS

A distinguishing feature of the present model for the mixing length is its strong dependence on the local velocity in the boundary layer, in contrast to the original formulation in terms of a local distance from the wall. A second feature following the first is the model's ability to incorporate, through a free constant, the surface roughness effect by virtue of an amplification factor. Numerically, the model behaves well and mimics quite faithfully the experimental features of the flow over rough surfaces. On the other hand, it is difficult to conclude whether the exponential form is more, or less conceptually acceptable than Prandtl's model since both are based on the empiricism of the wall law. From a calculational viewpoint, the exponential model eliminates the need for separate descriptions of the mixing length according to the range of the roughness Reynolds number.

The present model requires, as does any other model, that for each surface roughness pattern the relation between R and k^+ be known. Within the scope of this work, exhaustive compilation of the empirical R–k^+ relations is neither intended nor possible, for the simple fact that although there were a large number of rough surfaces tested, available information was not sufficient to be processed into a form suitable for the present model effort. Further research should include effort to explore how the model can be extended to include the effects of pressure gradient

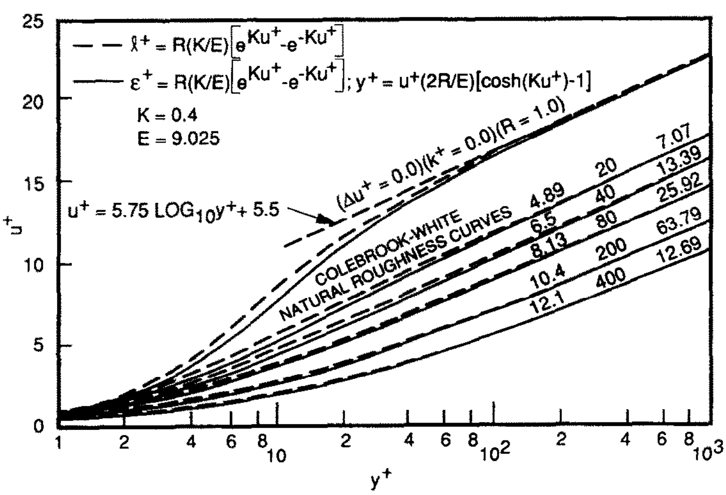

FIG. 6. Wall-region velocities for rough surfaces (Colebrook–White).

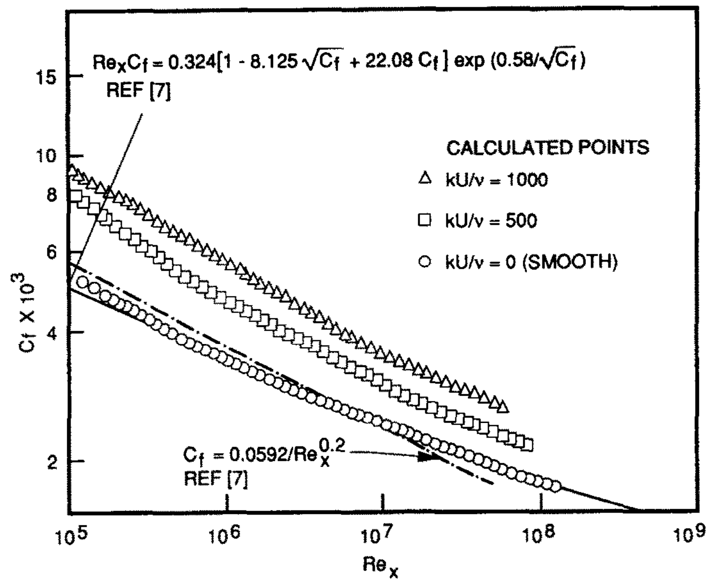

FIG. 7. Calculated local friction coefficients of flat plates, Nikuradse's sandgrain roughness, $K = 0.4$, $E = 7.768$.

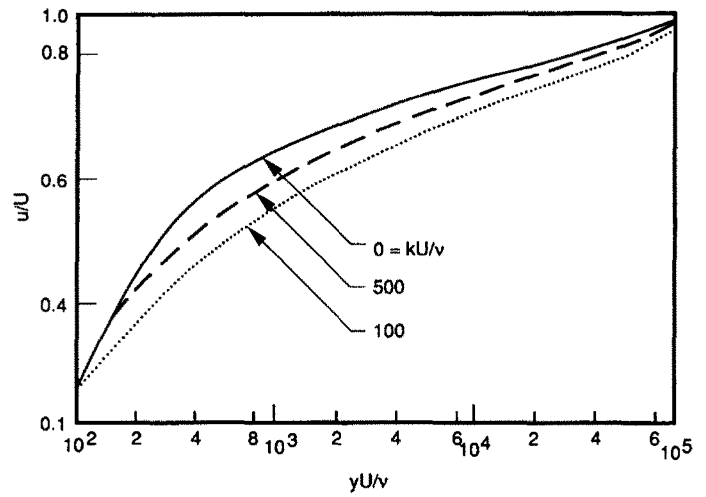

FIG. 8. Calculated rough-surface boundary layer velocity profiles at $Re_x = 10^7$, $K = 0.41$, $E = 7.768$.

and surface transpiration. Of course, extension to heat transfer with a thermal mixing length appears to be a natural sequel. The latter has been pursued and will be reported later.

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UN MODELE DE LONGUEUR DE MELANGE POUR LES COUCHES LIMITES TURBULENTES SUR DES SURFACES RUGUEUSES

Résumé—On développe une méthodologie pour analyser les couches limites turbulentes sur des surfaces rugueuses. Une nouvelle formulation du modèle de longueur de mélange exprimée en variable de vitesse est choisie et l'effet de la rugosité de surface est représenté par un facteur d'amplitude qui multiplie la longueur de mélange. Il est égal à l'unité pour une surface lisse et il est plus grand pour des surfaces rugueuses selon le type et l'amplitude des rugosités. La nouvelle méthode est essayée dans le cas d'un écoulement sur une plaque plane avec deux rugosités de surface. Avec un nombre de Reynolds de 10 millions, le frottement local pariétal est augmenté de 60% quand le rapport hauteur sur longueur de la rugosité est 1 : 10000.

EIN MISCHUNGSWEGMODELL FÜR TURBULENTE GRENZSCHICHTEN AN RAUHEN OBERFLÄCHEN

Zusammenfassung—Ein Verfahren zur Untersuchung turbulenter Grenzschichten an rauhen Oberflächen wird entwickelt. Eine Neuformulierung des Mischungswegmodells—ausgedrückt in der Geschwindigkeitsvariablen—ist grundlegend für die Erstellung des Modells. Der Einfluß der Oberflächenrauhigkeit ist in einem Verstärkungsfaktor als Multiplikator des Mischungswegs enthalten. Er steigt vom Wert 1 für eine glatte Oberfläche zu größeren Werten bei rauhen Oberflächen an, abhängig von Art und Größe der Rauhigkeit. Die neue Methode wird für den Fall einer Strömung über eine ebene Platte mit zwei unterschiedlichen Oberflächenrauhigkeiten dargestellt. Bei der längenbezogenen Reynolds-Zahl 10^7 wird der örtliche Reibungsbeiwert für das Verhältnis Rauhigkeitshöhe zu -länge 1 : 10000 um mehr als 60% erhöht.

МОДЕЛЬ ДЛИНЫ СМЕШЕНИЯ ДЛЯ ТУРБУЛЕНТНЫХ ПОГРАНИЧНЫХ СЛОЕВ НАД ШЕРОХОВАТЫМИ ПОВЕРХНОСТЯМИ

Аннотация—Разрабатывается методология анализа турбулентных пограничных слоев над шероховатыми поверхнохтями. При моделировании использована новая формулировка модели длины смешения, содержащая скорость, а эффект шероховатости поверхности включен в множитель при длине смешения. Он увеличивается от единицы (для гладких поверхностей) и зависит от вида и величины шероховатости. Применение нового метода илиютрируется на примере обтекания плоской пластины с двумя различными шероховатостями поверхности. Когда число Рейнольдса, основанное на длине, составляет 10 миллионов, локальный коэффициент трения увеличивается на 60% при отношении высоты шероховатости к длине, равном 1 : 10000.